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Note

Polynomial Series versus Sinc Expansions for Functions with Corner or Endpoint Singularities

In a review of methods that use "Whittaker cardinal" or "sinc" functions, Stenger [1] shows that these basis functions—in combination with a change-of-variable—are a powerful tool for approximating a function with *weak* singularities at the *ends* of the interval. Although Stenger himself is careful to note that the same optimal convergence rate can be obtained with other basis functions, he does not elaborate or give examples.

In this note, we show that the change-of-variable—not the use of sinc functions—is the key to success in coping with endpoint singularities. We explicitly construct approximations using *mapped* orthogonal polynomials[1] which have the property of "exponential" or "infinite order" convergence for $f(x)$ which has weak singularities at the endpoints but is regular on the interior of the interval. (We define what we mean by a "weak" singularity in Eq (1) below.) For simplicity, most of the analysis is limited to functions of one variable, but corner singularities for two-dimensional boundary value problems are a very important application which will be briefly discussed at the end.

We shall standardize the interval to $x \in [-1, 1]$ and assume $f(x)$ has singularities at the endpoints of the form

$$f(x) = (1 - x^2)^\alpha \, g(x) \tag{1}$$

where $g(x)$ is free of singularities on $[-1, 1]$ and where $\alpha > 0$ so that the $f(x)$ is bounded even at the branch points. (This boundedness of $f(x)$ is what we mean by a "weak" singularity.) If we expand $f(x)$ in a Chebyshev series, Elliott [4] has shown that the coefficients (for $\alpha > 0$) are

$$a_n \sim O(1/n^{1+2\alpha}). \tag{2}$$

This poor convergence—coefficients decreasing only as an *algebraic* function of

[1] Because of the mapping, the terms of the series will be *transcendental* functions of the original coordinate x. This is what makes it possible to evade the theorem that *polynomials* in x could not give better than algebraic convergence because of the singularities.

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n—is the best possible for *polynomials*, and has prompted a search for alternatives. The successful options use a series of terms which are *transcendental* functions of x.

Stenger [1] showed that it is possible to create an approximation through a three-step procedure whose error decreases *exponentially* rather than *algebraically*. The first step is to transform the interval $x \in [-1, 1]$ to $y \in [-\infty, \infty]$ through the mapping

$$x = \tanh(ky) \tag{3}$$

where k is an arbitrary scale factor. The second step is to choose a grid spacing h and then approximate $f(y[x])$ through the "sinc expansion" or "Whittaker cardinal" approximation, which is

$$f(y) \approx \sum_{j=-\infty}^{\infty} f(jh) \operatorname{sinc}([y - jh]/h); \qquad y \in [-\infty, \infty] \tag{4}$$

where the "sinc" function is defined by

$$\operatorname{sinc}(z) \equiv \sin(\pi z)/(\pi z). \tag{5}$$

The final step is to truncate the infinite series in (4) so that we sum over a finite number of grid points N. Stenger [1] goes on to show that the sinc expansion can be used to solve differential and integral equations, but for simplicity, we will assume $f(x[y])$ is a known function.

There are two sources of error in (4): a "grid-spacing" error because the interpolation points are a finite distance h apart and a "grid-span" error because we must truncate the infinite sum in (4), which implicitly restricts the grid points to some finite span of the interval in y. Stenger shows that minimizing the combined effects of these two sources of error requires taking $h \sim O(1/N^{1/2})$ where N is the total number of grid points. The error is then $O(\exp[-pN^{1/2}])$ for some constant p provided that

$$\alpha > 0, \tag{6}$$

which is the condition that $f(x)$ remain bounded at $x = \pm 1$.

However, the *key* to *defusing* the *branchpoints* is the *tanh-mapping*. If we substitute (3) into (1), we find

$$f(x[y]) = \operatorname{sech}^{2\alpha}(ky)\, g(x[y]) \tag{7}$$

where the lack of singularities in $g(x)$ on $[-1, 1]$ implies that $g(x[y])$ is bounded for all real y. Once the problem has been converted to the interval $y \in [-\infty, \infty]$, one can use *any* of a wide variety of alternatives to approximate the function, not just the sinc series (4).

Boyd [3] shows that for a function like (7), the expansion in Hermite functions, $\psi_n(y)$, will converge for all real y. As for the sinc series, the error is $O(\exp[-qN^{1/2}])$ for some constant q. In terms of the original coordinate x, we are escaping the ineffectiveness of a polynomial series by using the transcendental basis functions $\psi_n(y[x])$ instead.

A Hermite pseudospectral method would use a computational grid composed of the roots of the $(N+1)$-degree Hermite function, which are separated by an *average* grid spacing of $O(N^{-1/2})$. (For a given N, the distance between neighboring grid points is variable, but does not differ much from the average.) Stenger [1] shows that $h \sim O(N^{-1/2})$ is usually optimum for the sinc expansion as well. For either series, the mapping is crucial: the tanh function transforms a grid of points which is evenly (or almost evenly) spaced in y into a grid in x in which the gridpoints are clustered near the endpoints at $x = \pm 1$. The spacing of adjacent points in x decreases *exponentially* with N near the branch-points, which is precisely what is needed to defeat these singularities. In contrast, the Chebyshev pseudospectral method (without a mapping of the coordinate) would use a grid with a nearest-neighbor separation no smaller than $O(1/N^2)$; for any positive α, Eq. (2) shows that the highest calculated Chebyshev coefficient, a_N, is proportional to a negative power of N also. Thus, a *grid spacing* which decreases *algebraically* fast with N gives an *error* that decays *algebraically* with N, too. In contrast, decreasing the grid spacing in x *exponentially* fast near $x = \pm 1$ gives an error which decreases *exponentially* with N also in spite of the bounded endpoint singularities.

Estimating the constants p and q in error formulas like (7) is difficult in general, but it is instructive to consider the simple example

$$f(y) = \operatorname{sech}([\pi/2]^{1/2} y) \tag{8}$$

which is (7) for the special case $\alpha = 1/2$, $k = (\pi/2)^{1/2}$, and $g(y) \equiv 1$. This choice of the scale factor k optimizes the convergence of the Hermite series and we find from [3]

$$E_{\text{Hermite}}(N) \sim O(\exp[-1.772\, N^{1/2}]). \tag{9}$$

Using Theorem 2.1 of Lund and Riley [2] to optimize the sinc grid gives $h = (2\pi/N)^{1/2}$ where N is the total number of interpolation points and

$$E_{\text{sinc}}(N) \sim O(\exp[-1.5708\, N^{1/2}]). \tag{10}$$

Thus, the series of orthogonal polynomials is superior (at least asymptotically as $N \to \infty$) to employing the sinc expansion.

It would be wrong to conclude, however, that the sinc expansion is an *inferior* method. The difference between (9) and (10) is quite small, and sinc functions are simpler and easier to program than Hermite series. The point is rather that the sinc expansion is not the *only* choice.

Indeed, Boyd [5] is the basis for a third alternative for expanding functions with branch-points at the ends of the approximation interval. That earlier article discusses how problems on $y \in [-\infty, \infty]$ can be efficiently solved by mapping the infinite interval into $[-1, 1]$ and then applying Chebyshev polynomials. The map $x = \tanh(ky)$ is poor for this purpose because it will convert a function that decays exponentially in y into one with singularities at $x = \pm 1$ which spoil the convergence of the Chebyshev series as shown by (2).[2] It is amusing to see that the $\tanh(ky)$ map *is* useful when applied in the *opposite* direction to convert a function that really *is* singular at $x = \pm 1$ into one with exponentially convergent Hermite and sinc series. This suggests the third alternative for expanding this class of functions: employing the $\tanh(ky)$ map to transform from $[-1, 1]$ to $[-\infty, \infty]$ and then using an *algebraic* map of the kind described in [5] to transform back to $[-1, 1]$; the *doubly-mapped* Chebyshev polynomials will give exponential convergence. (Without the change of coordinates, of course, the Chebyshev series would converge algebraically with N.)

The class of functions with weak endpoint singularities may seem rather special, but Lund and Riley [2] give a number of one-dimensional examples. Partial differential equations are a very rich source of examples since boundary value problems may have singular solutions even if the equation is linear and constant coefficient. The classic illustration is

$$\nabla^2 u = -1 \tag{11}$$

on the square $[-1, 1] \times [-1, 1]$ with $u \equiv 0$ on all four sides of the domain. Strang and Fix [6] show that $u(x)$ has singularities of the form

$$u = r^2 \log(r) + \text{less singular terms} \tag{12}$$

in each corner where r is the radius of a local polar coordinate centered on the corner. Bowers and Lund [7] have reported success in solving problems like (10) using mapped sinc expansions, but the transformed Hermite series and the double-mapped Chebyshev polynomials would probably be equally effective.

The moral of this note is that there are many ways to deal with endpoint and corner singularities. The sinc series of Stenger is simple and exponentially convergent, but there are alternative basis functions that are just as good.

For all these pseudospectral methods, however, a change of coordinates which gives an exponential clustering of gridpoints near the singularities is essential.

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[2] There are some tricks that allow the tanh-mapping to be effective, at least occasionally, but the author is still investigating.

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JOHN P. BOYD

Department of Atmospheric and Oceanic Science
University of Michigan
2455 Hayward Avenue, Ann Arbor, Michigan 48109